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Studies in Nonlinear Stochastic Processes. III. Approximate Solutions of Nonlinear Stochastic Differential Equations Excited by Gaussian Noise and Harmonic Disturbances

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A fusion of the highly successful methods of harmonic and statistical linearization is used as a first approximation in determining, either iteratively or via a nonlinear integral equation, the effects of higher harmonics and non-Gaussian distortion terms on the second-order statistics of a wide variety of nonlinear stochastic differential equations perturbed by some linear combination of Gaussian noise and a periodic deterministic/stochastic excitation. Physical a posteriori applicability criteria are presented which justify when these higher order effects may be neglected. A simple modification of this statistical–harmonic linearization procedure based upon the Fokker–Planck variance is proposed.

KEY WORDS: Nonlinear stochastic differential equation; random noise; harmonic excitation.

1. INTRODUCTION

This article is the third in a series of papers[1,2] on the calculation of approximate second-order statistics for the certain class of nonlinear stochastic differential equations defined by

$$Q\left(\frac{d}{dt}\right)Y(t) + R\left(\frac{d}{dt}\right)x(t) = U\left(\frac{d}{dt}\right)F(t) \tag{1}$$

where $Q(d/dt)$, $R(d/dt)$, and $U(d/dt)$ are linear differential operators of time and $Y(t) = Y[x(t), \dot{x}(t),...]$ is some general nonlinearity which may be

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piecewise continuous or of Heaviside type. The first two papers, hereafter referred to as I and II, restricted the external excitation $F(t)$ to Gaussian noise. This report will extend the methodology developed in those previous articles to include excitations which are a linear combination of Gaussian and periodic deterministic or stochastic signals.

The approach taken in this paper is a fusion of the methods of statistical and harmonic linearization.[1–3] Harmonic linearization is an offshoot of the asymptotic analysis developed by Krylov and Bogoliubov (see Refs. 3 and 4) to study the response of nonlinear deterministic differential equations to periodic disturbances. By assuming that the steady state or stable limit cycle solution can be adequately approximated by a function periodic in the first harmonic, this linearization, as its name connotes, involves keeping at most the first harmonic in a Fourier expansion of the nonlinearity $Y(t)$. Thus, Eq. (1) can be decomposed into two linear equations, one characterizing the zeroth harmonic and the other the first harmonic. Both equations, however, are coupled via numerical Fourier coefficients, which formally are often nonlinear. Inasmuch as the effect of the forcing function is to initiate asymmetric oscillations with respect to the independent variables of the nonlinearity, these basic equations may be further simplified in the absence of this excitation, since even the zeroth harmonic may be discarded. On the other hand, they may become more complicated by introducing the higher harmonics.[4]

When $F(t)$ is some linear combination of Gaussian noise and a deterministic sinusoidal signal the statistics of the solution will exhibit purely Gaussian, periodic, and cross-modulation terms due to the interference between the random and harmonic components of the excitation. In a vein similar to the original harmonic and statistical linearization procedures, the combined statistical–harmonic linearization method assumes a solution in the form of the sum of two functions, one purely Gaussian and the other periodic in the first harmonic. By linearizing $Y(t)$ in accordance with this prescription, Eq. (1) again eventuates into a set of two linear equations, the first expressive of the Gaussian noise component and the second of the periodic component. Both equations are nonlinearly coupled via statistically calculated coefficients.

Although the rudiments of this technique have been applied with great success by a number of authors[4–7] to various specific problems, very little progress has been made to systematize the linearization procedures involved. This becomes transparent upon perusal of the pertinent literature and discovering the variety of heuristics invoked in the treatment of nonlinear systems exhibiting self-oscillations or when calculating the effective frequency for lightly damped, weakly anharmonic oscillator equations subjected to Gaussian excitations.

We therefore propose to reexamine the joint statistical–harmonic linearization scheme as a subset of a much more complicated, yet more exact approach. We begin in Section 2 with a brief description of a modified Krylov–Bogoliubov method due to Popov and Palitov,[4] whence harmonic linearization is derived and improved upon by the inclusion of higher harmonics.

In Section 3, we indicate what modifications of the harmonic and statistical linearization methods are necessary in order to study the approximate statistical response of system (1) to a combined input of Gaussian noise and a deterministic sinusoidal waveform. Physical a posteriori applicability criteria justifying linearization are discussed. An iterative scheme delimiting the influence of the higher harmonics on this approximation is derived in Section 4.

One can go one step beyond statistical–harmonic linearization by including in (1) the Gaussian distortion terms and terms in the higher harmonics arising due to the nonlinearity. This is illustrated in Section 5 with a specific example of an anharmonic oscillator $Y(t) = x^{2N+1}$, $N = 1$, in which the excitation $F(t)$ can either be the one considered previously in Section 3 or a linear combination of Gaussian noise and a random pulse (rectangular waves, sawtooth waves, etc.). The effect of including these higher order terms leads to an autocorrelation function equation of the quasilinear Green's function type.[1] This equation has been demonstrated in II to yield results superior to statistical linearization when $F(t)$ is Gaussian-delta-correlated noise. It is this author's contention that it will also be superior to statistical–harmonic linearization. The results of an extensive numerical computation for all the methods delineated here and their analog computer comparison will be the subject of a forthcoming paper.

2. HARMONIC LINEARIZATION OF NONLINEAR DETERMINISTIC DIFFERENTIAL EQUATIONS

Consider the nonlinear deterministic differential equation defined in (1) and suppose, for now, that $F(t)$ is the harmonic signal

$$F(t) = a_F^{(1)} \sin \Omega_F t \tag{2}$$

and, for simplicity, $Y(t) = f(x(t), \dot{x}(t))$,

$$R\left(\frac{d}{dt}\right)x(t) + Q\left(\frac{d}{dt}\right)f(x, \dot{x}) = U\left(\frac{d}{dt}\right)F(t) \tag{3}$$

The frequency Ω_F is a locked-in mode and the system in (3) responds by following this harmonic.

The conditions under which the stable periodic limit cycle solution $x = a \sin(\Omega_F t + \theta)$ exists may be determined by an asymptotic analysis due to Popov and Palitov.[4] This entails assuming a solution of the form

$$x(t) = x_0(t) + x_1(t) + \chi(t) \tag{4}$$

where $x_0(t)$, $x_1(t)$, and $\chi(t)$, respectively, denote the contributions of the zeroth, first, and higher harmonics resulting from the nonlinearity $f(x, \dot{x})$, with

$$x_1(t) = a \sin(\Omega_F t + \theta) \tag{5a}$$

$$\chi(t) = \sum_{k=2}^{n} \delta_k a \sin(k\Omega_F t + \theta_k) = \sum_{k=2}^{n} x_k(t) \tag{5b}$$

The function $x_0(t)$ reflects the asymmetry of the homogeneous equation response due to the external excitation $F(t)$ and/or asymmetry of the nonlinearity $f(x, \dot{x})$ around some origin. It may be set equal to zero if $F(t) \equiv 0$ or if $f(x, \dot{x})$ is a symmetric function of its independent variables. The coefficients $\delta_l = a_l/a$ ($\delta_l \ll 1$) in (5b) play the role of small parameters and ensure that the first harmonic is dominant in the solution of (3).

Our asymptotic analysis now proceeds in the following fashion. We examine the effects of $\chi(t)$ on $x(t)$ by performing a Taylor series expansion on $f(x, \dot{x})$,

$$f(x, \dot{x}) = f(x_0 + x_1, \dot{x}_0 + \dot{x}_1) + \left[\frac{\partial}{\partial x} f(x_0 + x_1, \dot{x}_0 + \dot{x}_1)\chi(t) + \frac{\partial}{\partial \dot{x}} f(x_0 + x_1, x_0 + x_1)\dot{\chi}(t)\right] + O(\delta_l^2) \tag{6}$$

which, when expressed in terms of a Fourier series expansion, becomes

$$f(x, \dot{x}) \sim f_0 + f_1 + \sum_{k=2}^{n} f_k + \sum_{k=0}^{n} g_k \tag{7}$$

where

$$f_0 = (1/2\pi) \int_0^{2\pi} f(x_0 + x_1, \dot{x}_0 + \dot{x}_1)\, d\psi \tag{8a}$$

$$f_1 = q(x_0, a) a \sin(\Omega_F t + \theta) + q'(x_0, a) a \cos(\Omega_F t + \theta) \tag{8b}$$

$$q = (1/\pi a) \int_0^{2\pi} f(x_0 + x_1, \dot{x}_0 + \dot{x}_1) \sin\psi\, d\psi \tag{8c}$$

$$q' = (1/\pi a) \int_0^{2\pi} f(x_0 + x_1, \dot{x}_0 + \dot{x}_1) \cos\psi\, d\psi \tag{8d}$$

with $\psi = \Omega_F t + \theta$ and where

$$f_k = q_k(x_0, a) a \sin(k\Omega_F t + \theta_k) + q_k'(x_0, a) a \cos(k\Omega_F t + \theta_k) \tag{9a}$$

$$q_k = (1/\pi a) \int_0^{2\pi} f(x_0 + x_1, \dot{x}_0 + \dot{x}_1) \sin(k\psi + \mu_k)\, d\phi \tag{9b}$$

$$q_k' = (1/\pi a) \int_0^{2\pi} f(x_0 + x_1, \dot{x}_0 + \dot{x}_1) \cos(k\psi + \mu_k)\, d\phi \tag{9c}$$

$$g_0 = (1/\pi a) \int_0^{2\pi} [(\partial/\partial x) f(x_0 + x_1, \dot{x}_0 + \dot{x}_1)\chi$$
$$+ (\partial/\partial \dot{x}) f(x_0 + x_1, \dot{x}_0 + \dot{x}_1)\dot{\chi}]\, d\phi \tag{10a}$$

$$g_k = B_k a \sin(k\Omega_F t + \theta_k) + C_k a \cos(k\Omega_F t + \theta_k) \tag{10b}$$

$$B_k = (1/\pi a) \int_0^{2\pi} [(\partial/\partial x) f(x_0 + x_1, \dot{x}_0 + \dot{x}_1)\chi$$
$$+ (\partial/\partial \dot{x}) f(x_0 + x_1, \dot{x}_0 + \dot{x})\dot{\chi}] \sin(k\psi + \mu_k)\, d\phi \tag{10c}$$

$$C_k = (1/\pi a) \int_0^{2\pi} [(\partial/\partial x) f(x_0 + x_1, \dot{x}_0 + \dot{x}_1)\chi$$
$$+ (\partial/\partial \dot{x}) f(x_0 + x_1, \dot{x}_0 + \dot{x}_1)\dot{\chi}] \cos(k\psi + \mu_k)\, d\phi \tag{10d}$$

with $\phi = k\Omega_F t + \theta_k$, $\mu_1 = 0$, and $\mu_k = \theta_k - k\theta$ $(k \geqslant 1)$.

Substitution of (4) and (7) into (3) yields a hierarchy of $n + 1$ linear equations, one for each of the $n + 1$ harmonics, which are all nonlinearly related via the Fourier coefficients q, q', q_k, q_k', B_k, and C_k,

$$R\left(\frac{d}{dt}\right) x_0 + Q\left(\frac{d}{dt}\right) [f_0 + g_0] = 0 \tag{11a}$$

$$R\left(\frac{d}{dt}\right) x_1 + Q\left(\frac{d}{dt}\right) \left[q(x_0, a) + \frac{q'(x_0, a)}{\Omega_F} \frac{d}{dt}\right] x_1$$
$$+ Q\left(\frac{d}{dt}\right) g_1 = a_F^{(1)} \sin \Omega_F t \tag{11b}$$

$$R\left(\frac{d}{dt}\right) x_k + Q\left(\frac{d}{dt}\right) [f_k + g_k] = 0 \tag{11c}$$

If the higher harmonics x_k are, as initially assumed, small in comparison to x_0 and x_1, (11c) may be disregarded and the defining relations for the zeroth and first harmonics will now be given by

$$R\left(\frac{d}{dt}\right) x_0 + Q\left(\frac{d}{dt}\right) f_0 = 0 \tag{12a}$$

$$R\left(\frac{d}{dt}\right) x_1 + Q\left(\frac{d}{dt}\right) \left[q(x_0, a) + \frac{q'(x_0, a)}{\Omega_F} \frac{d}{dt}\right] x_1 = a_F^{(1)} \sin \Omega_F t \tag{12b}$$

The Fourier components g_0 and g_1 have been dropped from Eqs. (12a) and (12b) since they incorporate the higher harmonics $k\Omega_F$ and are $O(\delta_l)$ with respect to f_0 and f_1.

Relations (12a) and (12b) are what is commonly known as *harmonic linearization.*

The conditions under which Eqs. (12a) and (12b) are valid have been termed the *applicability criteria* in I. Briefly, they state that the higher harmonics x_k will have a negligible effect in the solution of (3), even when the f_k are not small, $k \geqslant 2$, if

1. (a) $\deg \hat{Q}(i\Omega_F) < \deg \hat{R}'(i\Omega_F)$
   (b) $|\hat{Q}(ik\Omega_F)/\hat{R}'(ik\Omega_F)| \ll |\hat{Q}(i\Omega_F)/\hat{R}'(i\Omega_F)|, \quad k \geqslant 2$
   (c) $\lim_{k\to\infty} |\hat{Q}(ik\Omega_F)/\hat{R}'(ik\Omega_F)| \to 0 \quad \forall k$
2. The polynomials $R'(ik\Omega)$ cannot have purely real zeros, $k = 1, 2, 3, \ldots$. This criterion guarantees stability of the solution, with transients dying out as $t \to \infty$.[2]
3. The function $f(x, \dot{x})$ should have finite partial derivatives with respect to its independent variables x and $\dot{x}$, and should not be an explicit function of time. Thus $f(x, \dot{x})$ may belong to both the class of piecewise continuous and discontinuous functions of Heaviside type.

The polynomials $\hat{R}'(\cdot)$ and $\hat{Q}(\cdot)$ are the Fourier-transformed

$$R(\cdot) - \frac{a_F^{(1)}}{a}\left(\cos\theta - \frac{\sin\theta}{\Omega_F}\frac{d}{dt}\right) \qquad \text{and} \qquad Q(\cdot)$$

A lengthier discussion of these criteria and their consequences has been presented elsewhere.[3]

In the next section we show how harmonic and statistical linearization may be fused together in order to obtain the approximate second-order statistics for the nonlinear stochastic differential equation (1) in which $F(t)$ is some linear combination of Gaussian-delta-correlated noise and a deterministic periodic signal.

3. STATISTICAL AND HARMONIC LINEARIZATION

In the previous section we derived the harmonically linearized equations (12a) and (12b) for the zeroth and first harmonics of the stable solution to (3), the former, (12a), arising due to the imposition of the periodic excitation $F(t)$ on the homogeneous part of (3), and the latter, (12b), arising due to the postulated form of the solution. The higher harmonics were discarded since they were considered $O(\delta_l)$ in magnitude. Both equations were nonlinearly coupled through the coefficients q and q'.

If the excitation $F(t)$ is a linear combination of Gaussian noise $\xi(t)$ and a harmonic signal

$$F(t) = \xi(t) + a_F^{(1)} \sin \Omega_F t \tag{13}$$

[2] Criterion 2 was incorrectly stated in Ref. 1. This is the corrected version.

one may, in a vein similar to the above, postulate a stable solution of the form

$$x(t) = x_h(t) + x_\xi(t) = a\sin(\Omega_F t + \theta) + x_\xi(t) \tag{14}$$

where $x_h(t)$ is the harmonic component of the solution and $x_\xi(t)$ is the random component of the solution. To first order in the solution variables x_ξ and x_h we approximate the nonlinearity $f(x, \dot{x})$ as

$$f(x, \dot{x}) \sim h_1 x_h + h_2 x_\xi + h_3 \dot{x}_h + h_4 \dot{x}_\xi \tag{15}$$

When the applicability criteria are met, as is the case for wideband[3] random excitations, the coefficients $h_1, \dots, h_4$ may be found through the statistical linearization formulas,[(1)]

$$h_i = \langle f(x, \dot{x}) x_i \rangle / \sigma_{x_i}^2 \tag{16a}$$

where we have used the assumption that

$$\langle x_i x_j \rangle = \sigma_{x_i}^2\, \delta_{ij}, \qquad i, j = 1, \dots, 4 \tag{16b}$$

For convenience we have made the correspondence $(x_h, x_\xi, \dot{x}_h, \dot{x}_\xi) \to (x_1, x_2, x_3, x_4)$. Thus, with $\psi = \Omega_F t + \theta$,

$$\begin{aligned} h_1(a, \Omega_F, \sigma_x{}^2, \sigma_{\dot{x}}{}^2) &= (1/\pi a) \int_0^{2\pi} d\psi \sin\psi \\ &\quad \times \iint_{-\infty}^{+\infty} f(x_2 + a\sin\psi, x_4 + a\Omega_F \cos\psi) p(x_2) p(x_4)\, dx_2\, dx_4 \end{aligned} \tag{17a}$$

$$\begin{aligned} h_2(a, \Omega_F, \sigma_x{}^2, \sigma_{\dot{x}}{}^2) &= (1/2\pi\sigma_{x_2}^2) \int_0^{2\pi} d\psi \\ &\quad \times \iint_{-\infty}^{+\infty} x_2 f(x_2 + a\sin\psi, x_4 + a\Omega_F \cos\psi) p(x_2) p(x_4)\, dx_2\, dx_4 \end{aligned} \tag{17b}$$

$$\begin{aligned} h_3(a, \Omega_F, \sigma_x{}^2, \sigma_{\dot{x}}{}^2) &= (1/\pi a\Omega_F) \int_0^{2\pi} d\psi \cos\psi \\ &\quad \times \iint_{-\infty}^{+\infty} f(x_2 + a\sin\psi, x_4 + a\Omega_F \cos\psi) p(x_2) p(x_4)\, dx_2\, dx_4 \end{aligned} \tag{17c}$$

[3] For a given low-frequency filter with passband $0 \le \omega \le \omega_c$ a wideband excitation is one whose highest frequency $\omega_l \gg \omega_c$.

and

$$h_4(a, \Omega_F, \sigma_x{}^2, \sigma_{\dot{x}}{}^2)$$
$$= (1/2\pi\sigma_{x_4}^2)\int_0^{2\pi} d\psi$$
$$\times \iint_{-\infty}^{+\infty} x_4 f(x_2 + a\sin\psi, x_4 + a\Omega_F\cos\psi)p(x_2)p(x_4)\,dx_2\,dx_4 \qquad (17\text{d})$$

The averages taken in (17a)–(17d) are combined time and ensemble averages, with the former over the harmonic variable and the latter over the stochastic variable.

It is important to point out as a caveat lector that when a system fails to obey the applicability criteria, the use of formulas (16a) and (16b) may not give the best results for the second-order statistics of (3). Such is the case, in fact, for narrowband random excitations in which the effective linear part of the system $|\hat{Q}(i\Omega_F)/\hat{R}'(i\Omega_F)|$ is not much smaller than 1. Corresponding to this situation, the spectral density of the excitation may be either outside of the passband of the effective linear part of the system or much lower than the internal frequency of the effective linear part of the system. Statistical and/or harmonic linearization may still be utilized in the analysis of such systems, but not directly on Eq. (3). We will not go into these procedures in this paper, but refer the interested reader to Refs. 4 and 7.

Insertion of relations (13)–(15) into Eq. (3) leads to two linear equations nonlinearly coupled via the coefficients $h_1, \ldots, h_4$,

$$R\left(\frac{d}{dt}\right)x_\xi(t) + Q\left(\frac{d}{dt}\right)\left[h_2 + h_4\frac{d}{dt}\right]x_\xi(t) = U\left(\frac{d}{dt}\right)\xi(t) \qquad (18\text{a})$$

$$R\left(\frac{d}{dt}\right)x_h(t) + Q\left(\frac{d}{dt}\right)\left[h_1 + h_3\frac{d}{dt}\right]x_h(t) = U\left(\frac{d}{dt}\right)a_F^{(1)}\sin\Omega_F t \qquad (18\text{b})$$

Defining the transfer functions $\Phi_1(i\Omega_F)$ and $\Phi_2(i\Omega)$,

$$\Phi_1(i\Omega_F) = \hat{U}(i\Omega_F)/[(h_1 + ih_3\Omega_F)\hat{Q}(i\Omega_F) + \hat{R}(i\Omega_F)] \qquad (19\text{a})$$

$$\Phi_2(i\Omega) = \hat{U}(i\Omega)/[(h_2 + ih_4\Omega)\hat{Q}(i\Omega) + \hat{R}(i\Omega)] \qquad (19\text{b})$$

one can obtain from Eqs. (18b) and (18a), respectively, the amplitude and phase of $x_h(t)$ relative to the periodic component of $F(t)$ and the stationary autocorrelation function $R_{xx}(t)$,

$$a = |\Phi_1(i\Omega_F)|a_F^{(1)} \qquad (20\text{a})$$

$$\theta = \arg\Phi_1(i\Omega_F) \qquad (20\text{b})$$

$$R_{xx}(t) = R_{xx}^{(0)}(t) = (1/2\pi)\int_{-\infty}^{+\infty} e^{i\Omega t}|\Phi_2(i\Omega)|^2\,d\Omega \qquad (20\text{c})$$

4. THE EFFECTS OF HIGHER HARMONICS ON THE AUTOCORRELATION FUNCTION

When the applicability criteria described in the previous section do not apply to a particular system, i.e., for narrowband Gaussian noise whose spectrum lies outside of the passband of the effective linear part of the system, modification of statistical–harmonic linearization must be made in order for it to remain a viable solution technique. This can be done by the introduction of higher harmonics and has the added benefits in that the response to Gaussian noise and deterministic pulses and the effects of higher harmonic terms on self-oscillatory systems may be examined.

As in the previous sections, we begin our discussion with Eq. (3), but now $F(t)$ is a linear combination of harmonic signals and a not necessarily wideband Gaussian noise $\xi(t)$,

$$F(t) = \xi(t) + a_F^{(1)} \sin \Omega_F t + \sum_{l=2}^{m} a_F^{(l)} \sin(l\Omega_F t + \nu_l) \tag{21}$$

When the random disturbance $\xi(t)$ has a wideband spectrum, then, in accordance with the arguments presented in Section 3, the effective linear part of the system filters out the high-frequency components of the noise. To a first approximation (in the linearization sense) one can therefore assume that $\xi(t)$ couples with only the first harmonic of the deterministic signal and as a consequence seek a stable solution of (3) in the form

$$x(t) = x_\xi(t) + x_h(t) + \sum_{k=2}^{n} x_k(t), \qquad n \geqslant m \tag{22}$$

where $x_\xi(t)$ and $x_h(t)$ are the stochastic and deterministic processes defined in (14) and the $x_k(t)$ are the deterministic periodic functions defined in (5b).

We now combine the method of statistical linearization and the Popov–Palitov techniques which were discussed in the previous sections to approximate our nonlinearity $f(x, \dot{x})$ by

$$f(x, \dot{x}) \sim h_1 x_h + h_2 x_\xi + h_3 \dot{x}_h + h_4 \dot{x}_\xi + \sum_{k=2}^{n} f_k + g_1 \tag{23}$$

The first four terms of (23) are identical to those in (15) and represent the first harmonics and the purely Gaussian components of the nonlinearity. The statistically averaged coefficients $h_1, \ldots, h_4$ are given in Eqs. (17a)–(17d). The summation over the fifth term and the last term represent the higher harmonics in the Fourier series expansion $f(x, \dot{x})$ and its derivative and are defined by relations (7), (9), and (10b), but with $x_0 \equiv 0$. Higher order random terms arising due to the distortion of the Gaussian by the nonlinearity and analogous to the higher order harmonic terms have been neglected in this approximation of $f(x, \dot{x})$. They will be considered in Appendix A, with a sample computation for the Duffing oscillator performed in Section 5.

Substituting Eqs. (21)–(23) into (3) yields a hierarchy of $n + 1$ linear equations nonlinearly interrelated via the coefficients h_k, q_k, q_k', B_1, and C_1,[4]

$$R\left(\frac{d}{dt}\right)x_\xi(t) + Q\left(\frac{d}{dt}\right)\left[h_2 + h_4 \frac{d}{dt}\right]x_\xi(t) = U\left(\frac{d}{dt}\right)\xi(t) \tag{24a}$$

$$R\left(\frac{d}{dt}\right)x_h(t) + Q\left(\frac{d}{dt}\right)\left[h_1 + B_1 + \frac{h_3\Omega_F + C_1}{\Omega_F}\frac{d}{dt}\right]x_h(t)$$

$$= U\left(\frac{d}{dt}\right)a_F^{(1)} \sin \Omega_F t \tag{24b}$$

$$R\left(\frac{d}{dt}\right)x_k(t) + Q\left(\frac{d}{dt}\right)\delta_k^{-1}\left[q_k + \frac{q_k'}{k\Omega_F}\frac{d}{dt}\right]x_k(t)$$

$$= U\left(\frac{d}{dt}\right)a_F^{(l)} \sin(l\Omega_F t + \nu_l)\,\delta_{lk} \tag{24c}$$

Equation (24a) describes the time evolution of the noise component $x_\xi(t)$, whereas (24b) and (24c) describe, for each harmonic separately, the time evolution due to the periodic part of the solution. This system of equations may be solved iteratively to obtain the autocorrelation function $R_{xx}(t)$ from (24a) and the amplitudes a and a_k and phase shifts θ and θ_k from (24b) and (24c). Here δ_{lk} is the Kronecker delta. It should not be confused with the smallness parameter δ_k.

Computationally this is done as follows: We rephrase B_1, C_1, q_k, and q_k' as

$$B_1 = \sum_{j=2}^{n} (I_{j1}\,\delta_j \cos \mu_j + I_{j2}\,\delta_j \sin \mu_j) \tag{25a}$$

$$C_1 = \sum_{j=2}^{n} (I_{j3}\,\delta_j \cos \mu_j + I_{j4}\,\delta_j \sin \mu_j) \tag{25b}$$

$$q_k = r_k \cos \mu_k + s_k \sin \mu_k \tag{26a}$$

$$q_k' = s_k \cos \mu_k - r_k \sin \mu_k \tag{26b}$$

[4] We omit B_k, C_k, $k \geq 2$, from the remainder of this discussion since they are $O(\delta_l)$, the magnitude of their corresponding q_k and q_k'.

where

$$I_{j1} = (1/\pi) \int_0^{2\pi} \Psi_j(\psi) \sin \psi \, d\phi \tag{27a}$$

$$I_{j2} = (1/\pi) \int_0^{2\pi} \Theta_j(\psi) \sin \psi \, d\phi \tag{27b}$$

$$I_{j3} = (1/\pi) \int_0^{2\pi} \Psi_j(\psi) \cos \psi \, d\phi \tag{27c}$$

$$I_{j4} = (1/\pi) \int_0^{2\pi} \Theta_j(\psi) \cos \psi \, d\phi \tag{27d}$$

with

$$\Psi_j(\psi) = \frac{\partial}{\partial x} f(x_1, \dot{x}_1) \sin j\psi + \frac{\partial}{\partial \dot{x}} f(x_1, \dot{x}_1) j\Omega_F \cos j\psi \tag{28a}$$

$$\Theta_j(\psi) = \frac{\partial}{\partial x} f(x_1, \dot{x}_1) \cos j\psi - \frac{\partial}{\partial \dot{x}} f(x_1, \dot{x}_1) j\Omega_F \sin j\psi \tag{28b}$$

and where

$$r_k = (1/\pi a) \int_0^{2\pi} f(x_1, \dot{x}_1) \sin k\psi \, d\phi \tag{29a}$$

$$s_k = (1/\pi a) \int_0^{2\pi} f(x_1, \dot{x}_1) \cos k\psi \, d\phi \tag{29b}$$

By noting that

$$\begin{aligned} a_F^{(m)} \sin(m\Omega_F t + \nu_m) &= a_F^{(m)} \sin(m\Omega_F t + \theta_m - \theta_m + \nu_m) \\ &= \frac{a_F^{(m)}}{\delta_m a} \cos(\theta_m - \nu_m) x_m - \frac{a_F^{(m)} \sin(\theta_m - \nu_m) \dot{x}_m}{m \, \delta_m \, a\Omega_F} \end{aligned} \tag{30}$$

the harmonics of the external signal $F(t)$ may be expressed in terms of the corresponding harmonics of the sought periodic solution,

$$\begin{aligned} F(t) = {} & \frac{a_F^{(1)}}{a} \left(\cos \theta - \frac{\sin \theta}{\Omega_F} \frac{d}{dt} \right) x_h \\ & + \sum_{k=2}^{n} \frac{a_F^{(l)}}{\delta_l a} \left[\cos(\theta_k - \nu_l) - \frac{\sin(\theta_k - \nu_l)}{l\Omega_F} \frac{d}{dt} \right] x_k \, \delta_{kl} \end{aligned} \tag{31}$$

Thus, upon insertion of (31), Eqs. (24a)–(24c) can be written as

$$R\left(\frac{d}{dt}\right)x_\xi(t) + Q\left(\frac{d}{dt}\right)\left[h_2 + h_4\frac{d}{dt}\right]x_\xi(t) = U\left(\frac{d}{dt}\right)\xi(t) \tag{32a}$$

$$R\left(\frac{d}{dt}\right)x_h(t) + Q\left(\frac{d}{dt}\right)\left[h_1 + B_1 + \frac{h_3\Omega_F + C_1}{\Omega_F}\frac{d}{dt}\right]x_h(t)$$

$$= U\left(\frac{d}{dt}\right)\left[\frac{a_F^{(1)}}{a}\left(\cos\theta - \frac{\sin\theta}{\Omega_F}\frac{d}{dt}\right)\right]x_h(t) \tag{32b}$$

$$R\left(\frac{d}{dt}\right)x_k(t) + Q\left(\frac{d}{dt}\right)\delta_k^{-1}\left[q_k + \frac{q_k'}{k\Omega_F}\frac{d}{dt}\right]x_k(t)$$

$$= U\left(\frac{d}{dt}\right)\frac{a_F^{(l)}}{\delta_l a}\left[\cos(\theta_k - \nu_l) - \frac{\sin(\theta_k - \nu_l)}{l\Omega_F}\frac{d}{dt}\right]x_k(t)\,\delta_{kl} \tag{32c}$$

To initiate the iteration procedure, we neglect the contribution of (32c) and calculate a and θ from (18a) and (18b). These parameters are then inserted into (32c), which, upon Fourier transformation in the $k\Omega_F$-plane, yields

$$\hat{R}(ik\Omega_F) + \hat{Q}(ik\Omega_F)\,\delta_k^{-1}(q_k + iq_k')$$

$$= \hat{U}(ik\Omega_F)\frac{a_F^{(l)}}{\delta_l a}\left[\cos(\theta_k - \nu_l) - i\sin(\theta_k - \nu_l)\right]\delta_{kl} \tag{33}$$

Recognizing that

$$q_k + iq_k' = (r_k + is_k)e^{-i\theta_k}e^{ik\theta} \tag{34a}$$

$$\cos(\theta_k - \nu_k) - i\sin(\theta_k - \nu_k) = e^{-i\theta_k}e^{i\nu_k} \tag{34b}$$

we can transform (33) into

$$\delta_k e^{i\theta_k} = -\frac{\hat{Q}(ik\Omega_F)}{\hat{R}(ik\Omega_F)}(r_k + is_k)e^{ik\theta} + \frac{\hat{U}(ik\Omega_F)}{\hat{R}(ik\Omega_F)}\frac{a_F^{(l)}}{a}e^{i\nu_l}\,\delta_{lk} \tag{35}$$

which subsequently yields for the amplitudes and phase shifts δ_k and θ_k, respectively,

$$\delta_k = \left|-\frac{\hat{Q}(ik\Omega_F)}{\hat{R}(ik\Omega_F)}(r_k + is_k)e^{ik\theta} + \frac{\hat{U}(ik\Omega_F)}{\hat{R}(ik\Omega_F)}\frac{a_F^{(l)}}{a}e^{i\nu_l}\,\delta_{lk}\right| \tag{36a}$$

$$\theta_k = \arg\left[-\frac{\hat{Q}(ik\Omega_F)}{\hat{R}(ik\Omega_F)}(r_k + is_k)e^{ik\theta} + \frac{\hat{U}(ik\Omega_F)}{\hat{R}(ik\Omega_F)}\frac{a_F^{(l)}}{a}e^{i\nu_l}\,\delta_{lk}\right] \tag{36b}$$

For the particular case in which $F(t)$ is a purely sinusoidal plus Gaussian excitation, i.e., $a_F^{(l)} = 0$, $l \geqslant 2$,

$$\delta_k = |\hat{Q}(ik\Omega_F)/\hat{R}(ik\Omega_F)|(r_k^2 + s_k^2)^{1/2} \tag{37a}$$

$$\theta_k = \arg[-\hat{Q}(ik\Omega_F)/\hat{R}(ik\Omega_F)] + \tan^{-1}(s_k/r_k) + k\theta \tag{37b}$$

By substituting the values of δ_k and θ_k from (36a) and (36b) into the equation for the first harmonic (32b), one is able to obtain a more accurate determination for a and θ, a' and θ'. This refined amplitude and phase shift may be resubstituted into (36a) and (36b) to produce new amplitudes and phase shifts δ_k' and θ_k'. This iterative procedure may be continued to convergence. |

5. ON THE STATISTICS OF THE RESPONSE OF DUFFING'S EQUATION TO THE COMBINED EFFECT OF GAUSSIAN-DELTA-CORRELATED NOISE AND A HARMONIC SIGNAL

In this section we shall determine an expression for the autocorrelation function $R_{xx}(t)$ for the Duffing anharmonic oscillator equation perturbed by the external excitation depicted in (13),

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \beta x^3 = \xi(t) + a_F \sin \Omega_F t \tag{38}$$

The details and their generalization to a random excitation of the form presented in (21) are given in Appendix A.

The procedure employed will be similar to the quasilinear Green's function method discussed in I and will therefore exhibit the non-Gaussian distortion terms, in contradistinction to the analysis of Section 4, introduced by the nonlinearity x^3. Furthermore, since the excitation has a periodic component, harmonic distortion terms and noise-harmonic interference terms will also appear.

Utilizing the quasilinear Green's function method, $x(t)$ is transformed into a Green's function convolution equation

$$x(t) = \int_0^\infty G(\tau)\{F(t - \tau) + \phi[x(t - \tau)]\}\, d\tau \tag{39}$$

with stationary autocorrelation function

$$R_{xx}(t) = \iint_0^\infty d\tau_1\, d\tau_2\, G(\tau_1)G(\tau_2)[R_{FF}(\zeta) + R_{\phi F}(\zeta) + R_{F\phi}(\zeta) + R_{\phi\phi}(\zeta)] \tag{40}$$

In Eqs. (39) and (40) the Green's function $G(\tau)$, the error term $\phi[x(t)]$, and the independent variable ζ are respectively defined as

$$G(\tau) = \begin{cases} \dfrac{2e^{-\alpha\tau/2}}{(4\gamma^2 - \alpha^2)^{1/2}} \sin\left(\dfrac{(4\gamma^2 - \alpha^2)^{1/2}}{2}\,\tau\right), & \tau > 0 \\ 0, & \tau < 0 \end{cases} \tag{41a}$$

$$\phi[x(t)] = [h_1 x_h(t) + h_2 x_\xi(t) - \beta x^3] \tag{41b}$$

and

$$\zeta = t + \tau_2 - \tau_1 \tag{41c}$$

with the effective frequency γ in (41a) being

$$\gamma^2 = \omega_0{}^2 + h_1 + h_2 \tag{42}$$

and where the statistical linearization coefficients h_1 and h_2 are given by (17a) and (17b). Evaluation of these integrals for the cubic nonlinearity results in

$$h_1 = 3\beta(\tfrac{1}{4}a_F{}^2 + \sigma_x{}^2) \tag{43a}$$

$$h_2 = 3\beta(\tfrac{1}{2}a_F{}^2 + \sigma_x{}^2) \tag{43b}$$

which respectively represent the gain of the effective linear equations (18a) and (18b). Thus, in the statistical linearization approximation the effect of a harmonic signal on noise is to shift the dispersion.

In order to compute $R_{xx}(t)$ from (40), we shall need closed form expressions for $R_{\phi F}(\zeta)$, $R_{F\phi}(\zeta)$, and $R_{\phi\phi}(\zeta)$. Using the quasinormal assumption for these mixed autocorrelation functions,[1] one finds that

$$\begin{aligned} R_{F\phi}(\zeta) = R_{\phi F}(\zeta) &= h_1 R_{x_h x_h}(\zeta) + h_2 R_{x_\xi x_\xi}(\zeta) - R_{Fx^3}(\zeta) \\ &= \tfrac{1}{2} h_1 a_F{}^2 \cos \Omega_F \zeta + h_2 R_{x_\xi x_\xi}(\zeta) - R_{Fx^3}(\zeta) \end{aligned} \tag{44a}$$

$$\begin{aligned} R_{\phi\phi}(\zeta) = {}& h_1{}^2 R_{x_h x_h}(\zeta) + h_2{}^2 R_{x_\xi x_\xi}(\zeta) - 2\beta[h_1 R_{x_h x^3}(\zeta) + h_2 R_{x_\xi x^3}(\zeta)] \\ &+ \beta^2 R_{x^3 x^3}(\zeta) \end{aligned} \tag{44b}$$

Upon expanding $R_{Fx^3}(\zeta)$, $R_{x_h x^3}(\zeta)$, $R_{x_\xi x^3}(\zeta)$, and $R_{x^3 x^3}(\zeta)$ by formula (A.8) or (A.22), we obtain

$$R_{Fx^3}(\zeta) = h_1 R_{x_h x_h}(\zeta) + h_2 R_{x_\xi x_\xi}(\zeta) \tag{45a}$$

$$\beta R_{x_h x^3}(\zeta) = h_1 R_{x_h x_h}(\zeta) \tag{45b}$$

$$\beta R_{x_\xi x^3}(\zeta) = h_2 R_{x_\xi x_\xi}(\zeta) \tag{45c}$$

$$\begin{aligned} \beta^2 R_{x^3 x^3}(\zeta) = {}& h_1{}^2 R_{x_h x_h}(\zeta) + h_2{}^2 R_{x_\xi x_\xi}(\zeta) + \beta^2\{6R^3_{x_\xi x_\xi}(\zeta) \\ &+ \tfrac{1}{32} a_F{}^6 \cos 3\Omega_F \zeta + 18[R^2_{x_\xi x_\xi}(\zeta) R_{x_h x_h}(\zeta) \\ &+ R_{x_\xi x_\xi}(\zeta) \cdot \tfrac{1}{16} a_F{}^4 \cos 2\Omega_F \zeta]\} \end{aligned} \tag{45d}$$

$R_{F\phi}(\zeta)$ and $R_{\phi\phi}(\zeta)$ reduce to

$$R_{F\phi}(\zeta) = R_{\phi F}(\zeta) = 0 \tag{46a}$$

$$\begin{aligned} R_{\phi\phi}(\zeta) = \beta^2\{&6R^3_{x_\xi x_\xi}(\zeta) + \tfrac{1}{32}a_F{}^6 \cos 3\Omega_F\, \zeta \\ &+ 18[R^2_{x_\xi x_\xi}(\zeta)R_{x_h x_h}(\zeta) + R_{x_\xi x_\xi}(\zeta)\cdot\tfrac{1}{16}a_F{}^4 \cos 2\Omega_F\zeta]\} \end{aligned} \tag{46b}$$

The first two terms in (46b) reflect the distortion of the purely Gaussian and first harmonic components of the autocorrelation function due to the nonlinearity. The final two terms portray the effects of interference between the Gaussian noise and the harmonic disturbance.

Substituting Eqs. (46a) and (46b) into (40) finally results in the autocorrelation function[1]

$$\begin{aligned} R_{xx}(t) = R^{(0)}_{xx}(t) &+ \frac{R_{x_h x_h}(t)}{(\gamma^2 - \Omega_F{}^2)^2 + \alpha^2\Omega_F{}^2} + \beta^2 \left(\frac{a_F{}^6 \cos(3\Omega_F t)}{32[(\gamma^2 - 9\Omega_F{}^2)^2 + 9\alpha^2\Omega_F{}^2]}\right. \\ &+ 6\int_{-\infty}^{+\infty} dt'\, R^{(0)}_{xx}(t')\{R^3_{x_\xi x_\xi}(t - t') + 3[R^2_{x_\xi x_\xi}(t - t')R_{x_h x_h}(t - t') \\ &\left. + R_{x_\xi x_\xi}(t - t')\tfrac{1}{16}a_F{}^4 \cos 2\Omega_F(t - t')]\}\right) \end{aligned} \tag{47}$$

6. DISCUSSION

In the previous sections we have developed a systematic approximation scheme for the second-order statistics of nonlinear stochastic differential equations whose response contains random and quasiperiodic terms. This may come about, for example, for conservative systems perturbed by Gaussian noise and deterministic periodic excitations and for nonconservative systems perturbed solely by Gaussian noise, the latter exhibiting self-oscillations. This work is complementary to I and II, which treated nonconservative systems excited by Gaussian noise. Quasiperiodic response terms were therefore absent.

The framework of our approach rests upon a fusion of the methods of harmonic and statistical linearization, which as a first-order theory works quite well when the noise is wideband, i.e., Gaussian-delta-correlated. This is due to the damping out of the higher harmonics of the periodic signal and the high-frequency components of the random noise by the effective linear part of the system, $|\hat{Q}(i\Omega_F)/\hat{R}'(i\Omega_F)|$. In this case—subject to applicability criterion 2—it is therefore in order to assume a stable limit cycle solution which is periodic with respect to the first harmonic and of a purely Gaussian form without non-Gaussian distortion terms.

If the Gaussian noise spectrum is somewhat different from wideband, i.e., narrowband noise with spectrum lying outside of the passband of the effective linear part of the system, one can still utilize statistical–harmonic linearization, but not on the differential equation directly.[7] Modification of the procedures involved are necessary in order to extend the range of applicability to once again bring the frequency band of the noise within the passband of the effective linear part of the system. This often requires choosing an alternative reference frequency to Ω_F when assuming a solution of $x(t)$.

Presumably one can dispense with, or at least minimize, the contributions of these ad hoc methods with inclusion of higher harmonics and non-Gaussian distortion terms, but within the statistical–harmonic linearization paradigm. This is necessary in order to produce the closed form expressions (32) and (47). Convergence of the iterative scheme parallels that of the Popov–Palitov method. Convergence of the Hammerstein integral equation (47) has been discussed in I. Use of the latter expression (47) may permit us to relax applicability criterion 2 and study the transient behavior of unstable systems. It is also conceivable that further improvement on the basic statistical–harmonic linearization can be obtained by readjusting the variance via the Fokker–Planck method introduced in II. To actually determine whether these conjectures are in fact borne out necessitates some detailed numerical comparison on specific examples. At present work of this nature is being undertaken and will be presented in a future publication.

APPENDIX A. ON THE COMPUTATION OF THE STATIONARY TIME AUTOCORRELATION FUNCTION $R_{g(F)g(F)}(t)$

In this appendix we delineate two procedures for computing the autocorrelation function $R_{g(F)g(F)}(t)$ for the nonlinear random process $g(F(t))$ in which $F(t)$,

$$F(t) = \xi(t) + a\sin(\omega t + \theta) \tag{A.1}$$

is comprised of Gaussian noise $\xi(t)$ and a harmonic signal $a\sin(\omega t + \theta)$ of amplitude a and phase θ. The phase θ is either deterministic or uniformly distributed between $[0, 2\pi]$. The averages over the harmonic variable in Sections A1 and A2 are then either time or ensemble averages. Since time and ensemble averages over autocorrelation functions are respectively defined as

$$(\omega/2\pi)\int_0^{2\pi/\omega} F(a\sin(\omega\tau + \theta))F(a\sin[\omega(t + \tau) + \theta])\,d\tau$$

and

$$(1/2\pi)\int_0^{2\pi} F(a\sin(\omega\tau + \theta))F(a\sin[\omega(t + \tau) + \theta])\,d\theta$$

one can easily show that the use of either average will leave the appearance of all equations formally invariant.

The first approach has been discussed in Ref. 1 and is related to the expansion of probability density functions in terms of an orthogonal polynomial basis. The second method makes use of characteristic functions and is due to Rice.[8] While the former is the simplest to apply for random processes of the type characterized by (A.1), the latter is much more elegant, lends itself to simple physical interpretation, and is in principle generalizable to random processes of the type

$$F(t) = \xi(t) + \sum_{k=1}^{n} a^{(k)} \sin(\omega_k t + \theta_k) \tag{A.2}$$

The summand in (A.2) may be either random or deterministic. For the former we choose the $\{\theta_k\}$ as being comprised of n independent random phases uniformly distributed in the interval $[0, 2\pi]$. The frequencies ω_k may be commensurable or incommensurable. Formal equivalence between the time and ensemble average formulas in Section A3, however, will only hold when the ω_k are incommensurable. This should be clear from the multiplicative nature of the correlation function, since frequencies of the type $\omega_k \pm \omega_l \pm \omega_m \pm \cdots$ appear. Commensurable frequencies, i.e., such frequencies that occur when the summand (A.2) is a Fourier series, will not admit the simple separable formulas (A.28)–(A.30), since the frequencies arising in (A.30) are not independent. Thus, in order to keep the discussion of all three subsections in line with each other, we shall consider all phases as random variables.

A1. Probability Density Expansions

Let $g(F)$ be some odd, nonlinear, random function of F having zero mean, $\langle g(F) \rangle = 0$, and stationary time autocorrelation function

$$R_{g(F)g(F)}(t) = \int_{\mathscr{D}} g(F_1) g(F_2) P(F_1, F_2)\, dF_1\, dF_2 \tag{A.3}$$

defined over some domain $\mathscr{D}$. Here $p(F_1, F_2)$ is the joint probability density function of $F_1 \equiv F(\tau)$ and $F_2 \equiv F(t + \tau)$, which, due to the statistical independence of the Gaussian and harmonic random processes, may be decomposed into

$$p(F_1, F_2) = p(\xi_1, \xi_2) p(\eta_1, \eta_2) \tag{A.4}$$

with $\eta_1 \equiv a \sin\phi$, $\eta_2 = a \sin(\phi + \omega t)$, and $\phi = \omega\tau + \theta$.

The joint density functions $p(\xi_1, \xi_2)$ and $p(\eta_1, \eta_2)$ have orthogonal polynomial series expansions[1]

$$p(\xi_1, \xi_2) = \frac{1}{2\pi\sigma_{\xi_1}\sigma_{\xi_2}} \exp\left[-\frac{1}{2}\left(\frac{\xi_1{}^2}{\sigma_{\xi_1}^2} + \frac{\xi_2{}^2}{\sigma_{\xi_2}^2}\right)\right] \times \sum_{n=0}^{\infty} \frac{1}{n!} \rho_{\xi_1\xi_2}^n(t) H_{en}\left(\frac{\xi_1}{\sigma_{\xi_1}}\right) H_{en}\left(\frac{\xi_2}{\sigma_{\xi_2}}\right) \tag{A.5a}$$

$$p(\eta_1, \eta_2) = \frac{1}{\pi^2}(a^2 - \eta_1{}^2)^{-1/2}(a^2 - \eta_2{}^2)^{-1/2} \sum_{n=0}^{\infty} \epsilon_n T_n\left(\frac{\eta_1}{a}\right) T_n\left(\frac{\eta_2}{a}\right) \cos n\omega t \tag{A.5b}$$

where $H_{en}(x)$ and $T_n(x)$ are, respectively, Hermite and Tchebycheff polynomials with orthogonality relations

$$\int_{-\infty}^{+\infty} H_{en}(x) H_{em}(x) \exp(-x^2/2)\, dx = (2\pi)^{1/2} n!\, \delta_{mn} \tag{A.6a}$$

$$\int_{-1}^{+1} \epsilon_n T_n(x) T_m(x)(1 - x^2)^{-1/2}\, dx = \pi\, \delta_{mn} \tag{A.6b}$$

The variance $\sigma_x{}^2$, normalized autocorrelation function $\rho_{xx}{}^{(t)}$, and ϵ_n are defined by

$$\sigma_x{}^2 = \langle x^2(t) \rangle = R_{xx}{}^{(0)} \tag{A.7a}$$

$$\rho_{xx}{}^{(t)} = R_{xx}{}^{(t)} / \sigma_x{}^2 \tag{A.7b}$$

and

$$\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n \geqslant 1 \end{cases} \tag{A.7c}$$

Substitution of (A.4) and (A.5) into (A.3) then yields for the autocorrelation function $R_{g(F)g(F)}(t)$

$$\begin{aligned} R_{g(F)g(F)}(t) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-1}^{+1} \int_{-1}^{+1} d\xi_1\, d\xi_2\, d\eta_1\, d\eta_2\, g(F_1) g(F_2) p(\xi_1, \xi_2) p(\eta_1, \eta_2) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \epsilon_k \rho_{\xi_1\xi_2}{}^n(t) r_{1,nk} r_{2,nk} \cos k\omega t \\ &= \sum_{n=0}^{\infty} a_{1n} a_{2n} \rho_{\xi_1\xi_2}{}^n(t) \end{aligned} \tag{A.8}$$

where

$$r_{l,nk} = \frac{1}{\pi} \int_{-1}^{+1} d(\sin\theta)(1 - \sin^2\theta)^{-1/2} T_k(\sin\theta) \times \left(\frac{1}{(2\pi\sigma_\xi{}^2 n!)^{1/2}} \int_{-\infty}^{+\infty} d\xi\, g(F_l) H_{en}\left(\frac{\xi}{\sigma_\xi}\right) \exp\left(\frac{-\xi^2}{2\sigma_\xi{}^2}\right)\right), \quad l = 1, 2 \tag{A.9a}$$

and

$$a_{1n}a_{2n} = \sum_{k=0}^{\infty} \epsilon_k r_{1,nk} r_{2,nk} \cos k\omega t \tag{A.9b}$$

A2. Method of Characteristic Functions—Noise and Harmonic Signal

Characteristic function methods may also be used to find $R_{g(F)g(F)}(t)$. The analytics involved, however, depends upon the integrability of $g(F)$ and requires defining a linear transformation in the complex z plane,

$$g(F) = (1/2\pi)\int_{\mathbb{C}} H(iz)e^{izF}\, dz \tag{A.10}$$

which is to be evaluated over some contour $\mathbb{C}$.

For most cases of interest $g(F)$ and $H(iz)$ can be classified into the following categories:

(a) If $g(F)$ is an L_2 function, then $\mathbb{C}$ is any line parallel to the real axis which lies in the strip of analyticity of $H(iz)$. Therefore $H(iz)$ is the complex Fourier transform of $g(F)$,

$$H(iz) = \int_{-\infty}^{+\infty} g(F)e^{-izF}\, dF \tag{A.11a}$$

and

$$g(F) = \int_{-\infty+is}^{\infty+is} H(iz)e^{izF}\, dz \tag{A.11b}$$

(b) If $g(F)$ vanishes for $F < 0$, then $H(iz)$ is the Laplace transform of $g(F)$,

$$H(iz) = \int_{0}^{\infty} g(F)e^{-izF}\, dF \tag{A.12a}$$

and

$$g(F) = \int_{-is-\infty}^{-is+\infty} H(iz)e^{izF}\, dz \tag{A.12b}$$

(c) If $g(F)$ is a nonlinearity of polynomial type, $g(F) = F^{2N+1}$, such that $g(F) \to \infty$ as $F \to \pm\infty$, then $H(iz)$ may be computed by the generalized Fourier integral or the bilateral Laplace transform.

We write $g(F)$ as the disjoint union of two functions $g_+(F)$ and $g_-(F)$

$$g(F) = g_+(F) + g_-(F) \tag{A.13}$$

having the properties that

$$g_+(F) = \begin{cases} g(F), & F > 0 \\ 0, & F < 0 \end{cases} \tag{A.14a}$$

$$g_-(F) = \begin{cases} 0, & F > 0 \\ g(F), & F < 0 \end{cases} \tag{A.14b}$$

$H(iz)$ may therefore be similarly partitioned as

$$\begin{aligned} H(iz) &= H_+(iz) + H_-(iz) \\ &= \int_0^\infty g_+(F)e^{-izF}\,dF + \int_{-\infty}^0 g_-(F)e^{-izF}\,dF \end{aligned} \tag{A.15}$$

with $H_+(iz)$ and $H_-(iz)$ existing in their own strips of analyticity; for the former it is in the lower complex z plane and for the latter it is in the upper complex z plane. As a result of (A.15), $g(F)$ may be represented as

$$g(F) = \frac{1}{2\pi}\int_{-is-\infty}^{-is+\infty} H_+(iz)e^{izF}\,dF + \frac{1}{2\pi}\int_{is-\infty}^{is+\infty} H_-(iz)e^{izF}\,dF \tag{A.16}$$

For the purpose of the body of this paper the subsequent development will utilize (A.16) as the definition for $g(F)$. Substituting (A.16) into (A.3) yields

$$R_{g(F)g(F)}(t) = \frac{1}{4\pi^2}\int_{\mathbb{C}_+}\int_{\mathbb{C}_-} H_1(iz_1)H_2(iz_2)\gamma(z_1, z_2)\,dz_1\,dz_2 \tag{A.17}$$

The contours of integration $\mathbb{C}_+$ and $\mathbb{C}_-$ in the complex z plane run parallel to the real axis in such a manner so as to miss singularities along the imaginary s axis. Here $\gamma(z_1, z_2)$ is the characteristic function for the sum of the random variables F_1 and F_2,

$$\gamma(z_1, z_2) = \langle \exp i(F_1z_1 + F_2z_2)\rangle \tag{A.18}$$

which, due to the statistical independence of $\xi(t)$ and the harmonic process, factors into the product of characteristic functions $\gamma_\xi(z_1, z_2)$ and $\gamma_h(z_1, z_2)$,

$$\gamma(z_1, z_2) = \gamma_\xi(z_1, z_2)\gamma_h(z_1, z_2) \tag{A.19}$$

By noting that

$$\begin{aligned} \gamma_\xi(z_1, z_2) &= \exp[-\tfrac{1}{2}(\sigma_{\xi_1}^2 z_1^2 + \sigma_{\xi_2}^2 z_2^2 + 2\rho_{\xi_1\xi_2}\sigma_{\xi_1}\sigma_{\xi_2}z_1z_2)] \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(\sigma_{\xi_1}\sigma_{\xi_2}\rho_{\xi_1\xi_2})^n}{n!} D_n(z_1)D_n(z_2) \end{aligned} \tag{A.20}$$

where

$$D_n(z_i) = z_i^n \exp(-\sigma_{\xi_i}^2 z_i^2/2), \qquad i = 1, 2$$

and that

$$\gamma_h(z_1, z_2) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp(ia\{z_1 \sin(\omega\tau + \theta) + z_2 \sin[\omega(t + \tau) + \theta]\})$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp\{ia[z_1 \sin\phi + z_2 \sin(\phi + \omega t)]\}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp\{ia[z_1^2 + z_2^2 + 2z_1z_2 \cos\omega t \cos(\phi + \phi_0)]^{1/2}\}$$

$$= J_0(a[z_1^2 + z_2^2 + 2z_1z_2 \cos\omega t]^{1/2})$$

$$= \sum_{k=0}^{\infty} (-1)^k \epsilon_k J_k(az_1)J_k(az_2) \cos k\omega t \tag{A.21}$$

$$\epsilon_0 = 1, \qquad \epsilon_k = 2, \qquad k \geqslant 1$$

insertion of the series representations given in formulas (A.20) and (A.21) into (A.17) results in

$$R_{g(F)g(F)}(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\epsilon_k}{n!}(\sigma_{\xi_1}\sigma_{\xi_2}\rho_{\xi_1\xi_2})^n h_{1,nk}h_{2,nk} \cos k\omega t$$

$$= \sum_{n=0}^{\infty} a_{1n}a_{2n}\rho^n_{\xi_1\xi_2}(t) \tag{A.22}$$

The coefficients $h_{l,nk}$ and $a_{1n}a_{2n}$ are expressed as

$$h_{l,nk} = \frac{(i)^{n+k}}{2\pi}\int_{\mathrm{C}} H(iz_l)D_n(z_l)J_k(az_l)\, dz_l \tag{A.23a}$$

$$a_{1n}a_{2n} = \sum_{k=0}^{\infty} \epsilon_k \frac{(\sigma_{\xi_1}\sigma_{\xi_2})^n}{n!} h_{1,nk}h_{2,nk} \cos k\omega t \tag{A.23b}$$

with the product coefficient $a_{1n}a_{2n}$ numerically identical to that defined in (A.9b). Of special interest are the coefficients h_{01} and h_{10}, since they are directly related to the coefficients of statistical linearization h_1 and h_2

$$h_{01} = h_1 a/2 \tag{A.24a}$$

$$h_{10} = h_2 \tag{A.24b}$$

One may assign a physical interpretation to the series in (A.22) by rewriting it as

$$R_{g(F)g(F)}(t) = \sum_{m=0}^{5} R_{g(F)g(F)}(t) \tag{A.25}$$

in which each of the $R^{(m)}_{g(F)g(F)}(t)$ corresponds to:

(i) $R^{(0)}_{g(F)g(F)}(t) = h^2_{00}$: mean squared response.

(ii) $R^{(1)}_{g(F)g(F)}(t) = h^2_{10} R_{x_\xi x_\xi}(t)$: linear part of autocorrelation function due to the noise $\xi(t)$.

(iii) $R^{(2)}_{g(F)g(F)}(t) = 2h^2_{01} \cos \omega t$: linear part of autocorrelation function due to the harmonic process.

(iv) $R^{(3)}_{g(F)g(F)}(t) = \sum_{n=2}^{\infty} \frac{h^2_{n0}}{n!} R^n_{x_\xi x_\xi}(t)$: nonlinear distortion of the autocorrelation function due to the noise $\xi(t)$.

(v) $R^{(4)}_{g(F)g(F)}(t) = 2 \sum_{k=2}^{\infty} h^2_{0k} \cos k\omega t$: nonlinear distortion of the autocorrelation function due to the harmonic process.

(vi) $R^{(5)}_{g(F)g(F)}(t) = 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n!} R^n_{x_\xi x_\xi}(t) \cos k\omega t$: the effect of superposing the harmonic and noise processes (interference).

In particular, we consider as an example for the preceding the polynomial nonlinearity $g(F) = F^{2N+1}$, $N = 1, 2, \ldots$. In this case

$$H(iz) = (2N + 1)!/(iz)^{2(N+1)} \tag{A.26}$$

and the coefficients $h_{l,nk}$ are to be evaluated over the Hankel contour.[8] The statistical linearization coefficients h_{01} and h_{10} are therefore

$$h_{01} = (2N + 1)! \sum_{m=0}^{N} \frac{(a/2)^{2m+1}(\sigma_\xi{}^2/2)^{N-m}}{m!\,(m + 1)!\,(N - m)!} \tag{A.27a}$$

and

$$h_{10} = (2N + 1)! \sum_{m=0}^{N} \frac{(a/2)^{2m}(\sigma_\xi{}^2/2)^{N-m}}{(m!)^2(N - m)!} \tag{A.27b}$$

A3. Method of Characteristic Functions—Noise and Linear Combination of Harmonic Processes

The arguments of Section 2 can be easily generalized to compute $R_{g(F)g(F)}(t)$ when the random process $F(t)$ has the form given in (A.2). All the preceding formulas remain the same, but now, due to the statistical independence of all the random phases θ_k, the characteristic function (A.18) factors into the product of $n+1$ characteristic functions $\gamma_\xi(z_1, z_2), \gamma_{h1}(z_1, z_2),\ldots,$ $\gamma_{hn}(z_1, z_2)$,

$$\gamma(z_1, z_2) = \gamma_\xi(z_1, z_2) \prod_{k=1}^{n} \gamma_{h_k}(z_1, z_2) \tag{A.28}$$

Upon repeating the arguments of the last section, it is a simple matter to show that

$$R_{g(F)g(F)}(t) = \sum_{m=0}^{\infty} \tilde{a}_{1m}\tilde{a}_{2m}\bar{\rho}^{m}_{\xi_1\xi_2}(t) \tag{A.29}$$

where

$$\tilde{a}_{1m}\tilde{a}_{2m} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \frac{(\sigma_{\xi_1}\sigma_{\xi_2})^m}{m!} \tilde{h}_{1,mk_1\ldots k_n}\tilde{h}_{2,mk_1\ldots k_n} \prod_{i=1}^{n} \epsilon_{k_i} \cos k_i\omega_i t \tag{A.30a}$$

and

$$\tilde{h}_{l,mk_1\ldots k_n} = \frac{(i)^{m+\sum_{j=1}^{n} k_j}}{2\pi} \int_{C} H(iz_l)D_m(z_l) \prod_{j=1}^{n} J_{k_j}(a^{(j)}z_l)\, dz_l \tag{A.30b}$$

In the special case when the amplitudes $a^{(k)}$, $k \geqslant 2$, are assumed to be small in magnitude and in comparison with $a^{(1)}$, one may approximate the product $\prod_{j=2}^{n} J_{k_j}(a^{(j)}z_l)$ by using Laplace's asymptotic formula[9] for $J_{k_j}(a^{(j)}z_l)$

$$J_{p/2-1}(at) = \frac{(at/2)^{p/2-1}}{\Gamma(p/2)} \exp\left(-\frac{a^2t^2}{2p}\right) \tag{A.31}$$

Since the product over the J_{k_j} will become vanishingly small unless $k_j = 0$, $j = 2,\ldots, n$, $R_{g(F)g(F)}(t)$ may be suitably approximated with $\tilde{h}_{l,mk_1\ldots k_n}$ and $\tilde{a}_{1m}\tilde{a}_{2m}$ as

$$\tilde{h}_{l,mk_10\ldots0} = \frac{(i)^{m+k_1}}{2\pi} \int_{C} H(iz_l)D_m(z_l)J_{k_1} \exp\left\{-\frac{z_l^2}{4}\left[\sum_{j=2}^{n} (a^{(j)})^2\right]\right\} \tag{A.32a}$$

$$\tilde{a}_{1m}\tilde{a}_{2m} = \sum_{k_1=0}^{\infty} \epsilon_{k_1} \frac{(\sigma_{\xi_1}\sigma_{\xi_2})^m}{m!}\tilde{h}_{1,mk_10\ldots0}\tilde{h}_{2,mk_10\ldots0} \cos k_1\omega_1 t \tag{A.32b}$$

Statistical linearization coefficients corresponding to (A.24) are $\tilde{h}_{l,010\ldots0}$ and $\tilde{h}_{l,100\ldots0}$.

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